Existance and uniqueness in an inverse source problem for a one-dimensional time-fractional diffusion equation

Salih Tatar

Department of Mathematics, Zirve University, 27260, Gaziantep, Turkey

Abstract. In this study, an inverse source problem for a one-dimensional time-fractional diffusion equation is considered. An existence theorem based on the minimization of an error functional between the output data and the additional data is proved. Then it is showed that the unknown source function can be determined uniquely by an additional data $u(0, t), 0 \leq t \leq T$ using an auxiliary uniqueness result and the Duhamel’s principle.

Keywords. Time-fractional diffusion equation; Fractional derivative; Additional data; Duhamel’s principle; Existence and uniqueness.

AMS Subject Classifications. 45K05, 35R30, 65M32.

1. Introduction

This study deals with an inverse source problem for the following one-dimensional time-fractional diffusion problem:

$$
\begin{aligned}
\partial_{0+}^\alpha u(x,t) &= u_{xx}(x,t) - q(x)u(x,t) + f(x), \quad -1 < x < 1, \; 0 < t < T, \\
 u(-1,t) &= u(1,t) = 0, \quad 0 < t < T, \\
 u(x,0) &= u_0(x), \quad -1 < x < 1,
\end{aligned}
$$

(1)

where $0 < \alpha < 1$ is the fixed fractional order, $q(x) \in C[-1,1]$ represents a spatially dependent source (or sink) term in a bar, $f(x) \in C[-1,1]$ is time-independent source term and $\partial_{0+}^\alpha$ refers to the (left-sided) Caputo fractional derivative with respect to $t$ of order $\alpha$ which is defined as follows:

$$
\partial_{0+}^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{u'(s)}{(t-s)^\alpha} ds, 
$$

where $\Gamma(.)$ denotes to standard Gamma function. Thus the model (1) reproduce the canonical diffusion model. A classical solution of the problem (1) is called a function $u = u(x,t)$ defined in the domain $\Omega_T := [-1,1] \times [0,T]$ that belongs to the space $C(\overline{\Omega}_T) \cap W^1_1(0,T) \cap C^2_x(-1,1)$ and satisfies problem (1). By $W^1_1(0,T)$, the space of the functions $z \in C^1(0,T]$ such that $z' \in L(0,T)$ is denoted. As it is known, a direct problem aims to find a solution that satisfies given an ordinary or partial differential equation and related to initial&boundary conditions. In some problems, the main
ordinary or partial differential equation and the initial&boundary conditions are not sufficient to obtain the solution, but, instead some additional conditions are required. Such problems are called the inverse problems. Inverse source problems are the problems that consist of finding the unknown source term via an additional data. These additional data may be given in an interior point, on the boundary, on the final time or on the whole domain. The inverse problem here consists of determining the unknown source term \( f(x) \) via the additional data \( u(0, t), 0 \leq t \leq T \) in the one-dimensional time-fractional diffusion problem (1). In this context, for the given inputs \( \alpha, q(x), f(x) \) and \( u_0(x) \), the problem (1) is referred to the direct problem. There are many works on the direct problem for fractional diffusion equations such as an initial&boundary value problem [1-7, also see references therein]. In the literature, to the best knowledge of the author, there are few works on inverse source problems for fractional diffusion equations in spite of their physical and practical importance [8-9]. Moreover there is no study on the existence in inverse source problems related to fractional differential equations. In order to fill this gap, this study investigates existence and uniqueness of the source term \( f(x) \) via the additional data \( u(0, t), 0 \leq t \leq T \) in the one-dimensional time-fractional diffusion problem (1).

The remainder of this paper comprises three sections: In the next section, some preliminaries are given for completeness of the content. Main result of the paper is stated and proved in the section 3. The final section of the paper contains discussions and comments on planned studies.

2. Preliminaries

This section is devoted to some usefull lemmas related to solution of the direct problem (1), the Duhamel’s principle, the fractional-order derivative in the Caputo sense and the Mittag-Leffler function.

**Lemma 2.1.** ([10], page 100) Let \( D^\alpha_{a+}u \) be the fractional-order derivative in the Riemann-Liouville sense, namely

\[
D^\alpha_{a+}u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_a^t \frac{u(s)}{(t-s)^\alpha} ds.
\]

Then \( D^\alpha_{a+}u \) exist almost everywhere for \( u \in AC(I) \) where \( I = (a, b) \) and \( AC(I) \) is the space of all absolutely continuous functions on \( I \). Furthermore, \( D^\alpha_{0+}u \in L_r(I), 1 \leq r \leq 1/\alpha \) and the following relation holds:

\[
D^\alpha_{0+}u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^\alpha} + D^\alpha_{a+}u(t).
\]

It is well known that the classical Duhamel principle reduces the Cauchy problem for an inhomogeneous partial differential equation to the Cauchy problem for the corresponding homogeneous equation. In [11], the authors extend this famous principle to a wide class of fractional order differential operator equations. Define \( A : D \rightarrow L^2[-1, 1] \) by \( Au := -u_{xx}(x, t) + q(x)u(x, t) \) where \( D = C(\overline{\Omega}) \cap W^1_t(0, T) \cap C^2_x(-1, 1) \). In this
context the following lemma is correct since the operator $Au := -u_{xx}(x,t) + q(x)u(x,t)$ is a closed linear operator in the domain $\mathbb{D}$:

**Lemma 2.2.** ([11]) (Duhamel’s principle) Let $v(x,t)$ be the solution of the following problem:

$$
\begin{align*}
\begin{cases}
\partial_t^\alpha v(x,t) &= v_{xx}(x,t) - q(x)v(x,t) + f(x,t), & -1 < x < 1, 0 < t < T, \\
v(-1,t) &= v(1,t) = 0, & 0 < t < T, \\
v(x,0) &= 0, & -1 < x < 1.
\end{cases}
\end{align*}
$$

Then $v(x,t) = \int_0^t V(x,t;s)ds$ where $s > 0$ is a parameter, $t > s$ and $V(x,t;s)$ is the solution of the following problem:

$$
\begin{align*}
\begin{cases}
\partial_t^\alpha V(x,t; s) &= V_{xx}(x,t; s) - q(x)V(x,t; s), & -1 < x < 1, 0 < t < T, \\
V(-1,t; s) &= V(1,t; s) = 0, & 0 < t < T, \\
V(x,t; s)|_{t=s} &= D_0^\alpha f(x,s), & -1 < x < 1.
\end{cases}
\end{align*}
$$

**Lemma 2.3.** ([7]) The formal solution of the one dimensional time-fractional diffusion problem (1) is given in the following form:

$$
\begin{align*}
\begin{cases}
\partial_t^\alpha u(x,t) &= Au(x,t) + f(x,t), & t > 0, \\
u(x,0) &= u_0.
\end{cases}
\end{align*}
$$

and $\rho_n = \|X_n\|^{-2}$. Then, for each $\psi \in L^2(-1,1)$, we have the eigenfunction expansion

$$
\psi = \sum_{n=1}^{\infty} \rho_n \langle \psi, X_n \rangle X_n.
$$

We set

$$
U(\tau)u_0 = \sum_{n=1}^{\infty} \rho_n \langle u_0, X_n \rangle E_{\alpha,\alpha}(\tau^\alpha)X_n(x;q),
$$

and

$$
V(\tau)u_0 = \tau^{\alpha-1} \sum_{n=1}^{\infty} \rho_n \langle u_0, X_n(x) \rangle E_{\alpha,\alpha}(\tau^\alpha)X_n(x;q), \ t \geq 0.
$$

Then the solution to

$$
\begin{align*}
\begin{cases}
\partial_t^\alpha u(t) &= -Au(t) + f(t), & t > 0, \\
u(0) &= u_0,
\end{cases}
\end{align*}
$$

is

$$
\begin{align*}
\begin{cases}
\partial_t^\alpha u(t) &= \sum_{n=1}^{\infty} \rho_n \langle u_0, X_n \rangle E_{\alpha,\alpha}(\tau^\alpha)X_n(x;q), \\
u(0) &= u_0.
\end{cases}
\end{align*}
$$
is given by
\[ u(t) = U(t)u_0 + \int_0^t V(\tau - s)f(s)ds, \quad t > 0. \]  
(6)

The above formula will be used to prove existence of the solution.

**Lemma 2.4.** ([10], page 35) There exists a constant \( C_0 > 0 \) for \( \alpha < 2 \) and \( \pi \alpha /2 < \mu < \min\{\pi, \pi \alpha\} \) such that
\[ |E_\alpha(z)| \leq \frac{C_0}{1 + |z|^\mu}, \quad \mu \leq |\arg(z)| \leq \pi. \]

**Lemma 2.5.** ([12]) Let \( q \in M_w := \{ q \in L^\infty(0, 1) : q \leq 0, \|q\| \leq w \} \). Then the eigenvalues \( \lambda_n(q) \) and eigenfunctions \( X_n(x; q) \) to the Sturm-Liouville problem (5) satisfy the following estimates:
\[ \lambda_n(q) \geq n^2 \pi^2, \quad |\lambda_n(q_1) - \lambda_n(q_2)| \leq C\|q_1 - q_2\|_\infty, \quad \|X_n(x; q_1) - X_n(x; q_2)\|_\infty \leq \frac{C}{n}\|q_1 - q_2\|_\infty. \]

3. **An existence and uniqueness theorem for a solution of the inverse source problem**

In this section, we state and prove the main result of the paper. We apply the operator-theoretic approach given in [13] to prove the existence. For any \( f(x) \in C[-1, 1] \), denote a unique solution of the direct problem by \( u[f](x; t) \). We can obtain the output data \( u[f](0; t) \) for \( f(x) \in C[-1, 1] \) by using the formula (4). An optimal idea for solving the inverse problem is to minimize an error functional between the output data and the additional data. For a given target function \( \varphi \in L^2(-1, 1) \), we define the following minimization problem:

\[ \min_{f \in C[-1, 1]} \|u[f](0; t) - \varphi\|_{L^2(0, T)}^2. \]  
(7)

We set \( G[f](t) = u[f](0; t), \quad 0 < t < T \). Then we have the following lemma:

**Lemma 3.1.** \( G : C[-1, 1] \longrightarrow L^2(0, T) \) satisfies
\[ \|G[f] - G[\hat{f}]\|_{L^2(0, T)} \leq L\|f - \hat{f}\|_{C(0,1,1)}. \]

That is, \( G : C[-1, 1] \longrightarrow L^2(0, T) \) is Lipschitz continuous.

**Proof.** Let \( f, \hat{f} \in C[-1, 1], \ u = u[f], \ v = v[\hat{f}] \) and \( y = u - v \). It can be easily verified that \( y \) is a solution of the following problem:

\[
\left\{ \begin{array}{l}
\partial^\alpha_0 y(t) = -Ay(t) + f(x) - \hat{f}(x), \quad t > 0, \\
y(0) = 0.
\end{array} \right. \]
(8)

For arbitrarily fixed sufficiently \( \varepsilon > 0 \) and \( \delta > 0 \), we can define a fractional power \((-A + \delta)^{1+\frac{\varepsilon}{\alpha}}\) [14]. Moreover,
\[
\left\| (A + \delta)^{1+\frac{\varepsilon}{\alpha}} \right\| = \left\| \sum_{n=1}^{\infty} (\lambda_n + \delta)^{2+\frac{2\varepsilon}{\alpha}} \rho_n \langle z, X_n \rangle \right\| \frac{1}{2} < \infty.
\]
If we apply (6) to 9, we have
\[(A + \delta)^{\frac{1}{4} + \epsilon} y(t) = \int_0^t (A + \delta)^{\frac{1}{4} + \epsilon} V(\tau - s) \times (f(x) - \hat{f}(x)) ds, \quad t > 0.\]

By using Lemma 2.4 and Parseval equality, we have
\[\left\| (A + \delta)^{\frac{1}{4} + \epsilon} V(\tau) z \right\|_{L^2(0,T)} = \left\| \sum_{n=2}^{\infty} \tau^{\alpha-1} \rho_n \langle z, X_n \rangle E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) (\lambda_n + \delta)^{\frac{1}{4} + \epsilon} X_n \right\|_{L^2(0,T)} \]
\[\leq C_0 \left( \sum_{n=1}^{\infty} \tau^{2\alpha-2} \rho_n \langle z, X_n \rangle^2 \left( \frac{1}{1 + \lambda_n \tau^\alpha} \right)^2 \lambda_n^{\frac{1}{2} + 2\epsilon} \right)^{\frac{1}{2}} \]
\[\leq C_0 \left( \tau^{2\alpha-2} \tau^{-\left(\frac{1}{4} + 2\epsilon\right)\alpha} \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \rho_n \langle z, X_n \rangle^2 \left( \frac{\lambda_n \tau^\alpha}{1 + \lambda_n \tau^\alpha} \right)^{\frac{1}{2} + \epsilon} \right)^{\frac{1}{2}} \]
\[\leq C_0 \tau^{\left(\frac{3}{4} - \epsilon\right)\alpha - 1} \| z \|_{L^2(0,T)}. \]

Finally, noting that \(\epsilon > 0\) is sufficiently small and by (10), we obtain
\[\| y(t) \|_{L^2(0,T)} \leq C_0 \int_0^t (\tau - s)^{\left(\frac{1}{4} - \epsilon\right)\alpha - 1} \| f - \hat{f} \| ds \]
\[\leq L \| f - \hat{f} \|_{C([-1,1])}.\]

We note that the integral \(\int_0^t (\tau - s)^{\left(\frac{3}{4} - \epsilon\right)\alpha - 1} ds\) is finite. Thus, the proof is completed. \(\Box\)

An application of the usual argument on the compactness of the space \(C[-1,1]\) gives the following existence theorem.

**Theorem 3.1.** There exists \(f^* \in C[0,1]\) such that
\[\| u[f^*](0, t) - \varphi \|_{L^2(0,T)} \leq \| u[f](0, t) - \varphi \|_{L^2(0,T)},\]
for all \(f \in C[-1,1]\).

Now we prove a uniqueness theorem. We use the method given in [9] to prove the uniqueness. We note that the first sum in (4) corresponds to the solution of the one-dimensional homogeneous time-fractional diffusion equation with homogeneous boundary conditions and inhomogeneous initial condition; the second sum in (4) corresponds to the solution of the one-dimensional inhomogeneous time-fractional diffusion equation with homogeneous boundary and initial conditions, i.e., \(u = w + v\) where \(v\) is the solution of the problem (2) and \(w\) is the solution of the following problem:
\[\begin{cases}
\partial_t^\alpha w(x,t) = w_{xx}(x,t) - q(x)w(x,t), & -1 < x < 1, \quad 0 < t < T, \\
w(-1,t) = w(1,t) = 0, & 0 < t < T, \\
w(x,0) = u_0(x), & -1 < x < 1.
\end{cases}\]

5
It is very easy to see that the solution of the problem (10) is independent of the source function \( f(x) \). This means that the uniqueness question related to the inverse problem can be reduced to the following one: Can we use the additional data \( v(0,t), 0 \leq t \leq T \) to uniquely determine the unknown source term \( f(x) \) in the problem (2). Now we prove the following auxiliary uniqueness result.

**Lemma 3.2.** Let \( w_i, \ i = 1, 2 \) be the solution of the following problems:

\[
\begin{align*}
\frac{\partial^\alpha}{\partial x^\alpha} w_i(x,t) &= w_{ixx}(x,t) - q(x)w_i(x,t), \quad -1 < x < 1, \ 0 < t < T, \\
w_i(-1,t) &= w_i(1,t) = 0, \ 0 < t < T, \\
w_i(x,0) &= \varphi_i(x), \quad -1 < x < 1.
\end{align*}
\]

(11)

If \( w_1(0,t) = w_2(0,t), \ 0 \leq t \leq T \) then \( \varphi_1(x) = \varphi_2(x) \).

**Proof.** The eigenfunctions of the operator \( Au := -u_{xx}(x,t) + q(x)u(x,t) \) form a complete and orthogonal basis in \( L^2(-1,1) \). Then the initial functions in (11) can be written as

\[
\varphi_i(x) = \sum_{n=0}^{\infty} \varphi_{i,n}(x)X_n(x;q), \ i = 1, 2.
\]

(12)

Then noting that \( X_n(0;q) = 1 \) and by \( w_1(0,t) = w_2(0,t), \ 0 \leq t \leq T \) we obtain

\[
\sum_{n=0}^{\infty} \varphi_{1,n}E_\alpha(-\lambda_n t^\alpha) = \sum_{n=0}^{\infty} \varphi_{2,n}E_\alpha(-\lambda_n t^\alpha), \ 0 \leq t \leq T.
\]

(13)

By Lemma 2.4, Riemann-Lebesgue theorem and unique continuation for real analytic function we have

\[
\sum_{n=0}^{\infty} \varphi_{1,n}E_\alpha(-\lambda_n t^\alpha) = \sum_{n=0}^{\infty} \varphi_{2,n}E_\alpha(-\lambda_n t^\alpha), \ t > 0.
\]

(14)

We can take the Laplace transforms termwise on both sides of (14) to obtain

\[
\int_0^{\infty} e^{-zt}E_\alpha(-\lambda_n t^\alpha) \ dt = \frac{z^{\alpha-1}}{z^\alpha + \lambda_n}, \ \text{Re} \ z > 0.
\]

(15)

Furthermore, by Lemma 2.4, Lemma 2.5 and \( \varphi_{1,n} \to \infty \), we have

\[
\left| e^{-t\text{Re} z} \sum_{n=0}^{\infty} \varphi_{1,n}E_\alpha(-\lambda_n t^\alpha)X_n(x^*;q) \right| \leq C_0 e^{-t\text{Re} z} \left( \varphi_{1,0} + \sum_{n=1}^{\infty} \frac{\varphi_{1,n}}{\lambda_n t^\alpha \Gamma(1-\alpha)} \right)
\]

\[
\leq C_0 e^{-t\text{Re} z} \left( \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \right) \frac{1}{t^\alpha}
\]

\[
\leq C_0 e^{-t\text{Re} z} \left( \sum_{n=1}^{\infty} \frac{1}{n^2 \pi^2} \right) \frac{1}{t^\alpha}
\]

\[
\leq \frac{C_0}{t^\alpha} e^{-t\text{Re} z},
\]

(16)
where $C_0$ and $\hat{C}_0$ are positive constants. Also we note that $t^{-\alpha}e^{-t\text{Re} z}$ is integrable in $t \in (0, \infty)$ for fixed $z$ satisfying $\text{Re} z > 0$. According to the Lebesgue convergence theorem, (14) and (15) yield
\[
\int_0^\infty e^{-zt} \sum_{n=0}^\infty \varphi_{1,n} E_\alpha(-\lambda_n t^\alpha) dt = \sum_{n=0}^\infty \varphi_{1,n} \frac{z^{\alpha-1}}{z^\alpha + \lambda_n} = \sum_{n=0}^\infty \varphi_{2,n} \frac{z^{\alpha-1}}{z^\alpha + \lambda_n},
\]
which implies
\[
\sum_{n=0}^\infty \frac{\varphi_{1,n} - \varphi_{2,n}}{\rho + \lambda_n} = 0, \quad \text{Re} \rho > 0,
\]
(17)
where $\rho = z^\alpha$. Since we can analytically continue both sides of in $\rho$, the equality (17) holds for $\rho \in \mathbb{C} \setminus \{\lambda_n\}_{n \geq 0}$. We can find a suitable disk that contains 0 and does not contains $\{-\lambda_n\}_{n \geq 1}$. If we integrate (17) in this disk, the Cauchy integral formula yields $2\pi i(\varphi_{1,0} - \varphi_{2,0}) = 0$, which implies $\varphi_{1,0} = \varphi_{2,0}$. Repeating this argument, we obtain $\varphi_{1,n} = \varphi_{2,n}, \quad n = 1, 2, 3, \ldots$. Finally, by (12), we conclude that $\varphi_1(x) = \varphi_2(x)$. Thus, the proof is completed. \( \square \)

**Theorem 3.2.** The source function $f(x)$ can be determined uniquely by the additional condition $u(0, t), \; 0 \leq t \leq T$ in the problem (1).

**Proof.** By the Lemma 2.2, the solution $u(x, t)$ of the problem (1) can be written as summation of the solutions of the two different problems, namely $u(x, t) = V(x, t) + w(x, t)$ where $V(x, t)$ and $w(x, t)$ are the solutions of the problems (3) and (10) respectively. But we know that the solution of the problem (10) is independent of the source function. Then the inverse problem becomes more precise, i.e can we determine the source function $f(x)$ from $v(0, t), \; 0 \leq t \leq T$ in (3) uniquely. By using Lemma 3.2, we conclude that $D_0^\alpha f(x, s)$ is uniquely determined by $v(0, t)$. Then using Lemma 2.1 and the fact that the source function is time independent, we get the uniqueness of the source function $f(x)$ in the problem (3). Thus, the proof is completed. \( \square \)

4. Concluding remarks

In this paper, an existence and uniqueness theorem is proved for determination of the source function $f(x)$ in the problem (1) by the additional data $u(0, t), \; 0 \leq t \leq T$. The proof is based on the eigenfunction expansion of the solution to the initial/boundary value problem, spectral theory of the Sturm-Liouville operator and minimization of an error functional between the output data and additional data. The author of this paper plans to consider more general fractional inverse problems in upcoming studies. First, we plan to extend the existence and uniqueness result to general source functions, i.e., the source functions depend on both space and time variables. Furthermore, we plan to try to determine the unknown source function numerically. Later, we consider the inverse problems which aims to determine both order of the fractional time derivative and source function simultaneously. In this context, we investigate existence and unique identifiability of the solution and improve a
numerical method involves a regularization method. Also the inverse source/coefficient problems that contain space-time fractional diffusion equation are considered both theoretically and numerically. These are subjects of the future studies by the author of this paper.

References